University of California, Berkeley
Physics 110B, Fall 2004 (*Strovink*)
PROBLEM SET 4
Due at 5 PM on Wednesday, September 22, 2004

Here are problems 16-20, which should enhance your familiarity with relativistic kinematics:

16. (Surface muons)
"Surface" muon beams are important tools for investigating the properties of condensed matter samples as well as fundamental particles. Protons from a cyclotron produce π^+ mesons (quark-antiquark pairs) that come to rest near the surface of a solid target. The pion then decays to an (anti)muon (μ^+, a heavy electron-like particle) and a muon neutrino (ν_μ) via

$$\pi^+ \to \mu^+ + \nu_\mu \ .$$

Some of the muons can be captured by a beam channel and transported in vacuum to an experiment. In the limit that the mother pion decays at the surface of the target (so that the daughter muon traverses negligible material), the beam muons have uniform speed (and, as it turns out, 100% polarization as well). For the purposes of this problem, consider a muon to have $\frac{3}{4}$ of the rest mass of a pion; neglect the neutrino mass.
(a.)
Show that the surface muons travel at a speed which is a fraction $\beta_0 = \frac{7}{25}$ of the speed of light.
(b.)
If a muon's mean proper lifetime is τ, what fraction of the muons will decay during a flight path of length L in the laboratory? Express your answer in terms of β_0.

17.
In the lab frame $\mathcal{S}$, a particle with velocity $\beta c = \frac{4}{5}c$ decays into two massless particles with the same energy each.
(a.)
If the parent particle has mean (proper) lifetime τ in its *own* rest frame $\mathcal{S}'$, calculate its mean flight path L in the lab frame $\mathcal{S}$.
(b.)
In the lab frame $\mathcal{S}$, calculate the opening angle $\psi = \cos^{-1} \hat{p}_1 \cdot \hat{p}_2$ between the two daughter particles.

18.
Here's an adult version of Griffiths' Problem 12.35. In a pair annihilation experiment, a positron (mass m) with total energy $E = \gamma mc^2$ hits an electron (same mass, but opposite charge) at rest. (Griffiths has it the other way around, but that's unrealistic – it's easy to make a positron beam, but hard to make a positron target.) The two particles annihilate, producing two photons. (If only one photon were produced, energy-momentum conservation would force it to be a massive particle traveling at a velocity less than c.) If one of the photons emerges at angle θ relative to the incident positron direction, show that its energy ϵ is given by

$$\frac{mc^2}{\epsilon} = 1 - \sqrt{\frac{\gamma - 1}{\gamma + 1}} \cos\theta \ .$$

(In particular, if the photon emerges perpendicular to the beam, its energy is equal to mc^2, independent of the beam energy. Similar results have been used to design clever experiments.)

19.
If you have studied Rutherford scattering (elastic scattering of a nonrelativistic He nucleus from an Au nucleus), you have seen the *differential cross section* for this process written in the form

$$\frac{d\sigma}{d\Omega} \propto \frac{(zZe^2)^2}{\sin^4 \frac{\Theta}{2}} \ ,$$

where ze (Ze) is the electric charge of the He (Au) nucleus; Θ is the angle by which the He nucleus is elastically scattered, measured in the CM frame; and $d\Omega$ is an element of solid angle within which the He nucleus emerges in that frame. [A century ago, Rutherford-scattering data collected by graduate students who were used as particle detectors demonstrated that atoms contain charged point-like constituents (nuclei).] Here we revisit Rutherford scattering for *relativistic* particles.

Consider the 2+2 relativistic scattering process

$$p + a \rightarrow q + b \, ,$$

where p, a, q, and b denote both the particles and their 4-momenta. The *Mandelstam variables*, first written down by Berkeley emeritus professor Stanley Mandelstam, are

$$\begin{aligned} s &\equiv (p+a)\cdot(p+a) \equiv \text{CM energy}^2 \\ t &\equiv (q-p)\cdot(q-p) \equiv 4 \text{ momentum transfer}^2 \\ u &\equiv (b-p)\cdot(b-p) \equiv \text{cross channel transfer}^2 \end{aligned}$$

In this problem we are concerned with the Mandelstam variable t.

(a.)
Further assuming that the masses of particles p and q are negligible, show that

$$-t = 4\tfrac{E}{c}\tfrac{E'}{c}\sin^2\tfrac{\Theta}{2} \, ,$$

where E (E') is the energy of particle p (q), and Θ is the angle between $\vec{p}$ and $\vec{q}$.
(b.)
Why is $d(-t)$ a Lorentz invariant? $d\sigma$ is an area transverse to the beam direction. Why is $d\sigma$ invariant to Lorentz transformations along that direction? In a system of units where $\hbar = c = 1$, all quantities have dimensions that can be expressed in units of Joules. In those units, what are the dimensions of $d\sigma$?
(c.)
In part (b.) you showed that $d\sigma/d(-t)$ is a Lorentz invariant. If particle p (which becomes particle q) and particle a (which becomes particle b) both are structureless, and if the scattering is elastic (particles a and b both have the same mass), the only relevant Lorentz-invariant variable that is available to us is $-t$. On purely dimensional grounds, show that

$$\frac{d\sigma}{d(-t)} \propto \frac{1}{t^2} \, .$$

[If p (a) has electric charge ze (Ze), and they interact electromagnetically, the constant of proportionality is $4\pi z^2 Z^2 \alpha^2$, where the fine structure constant α is given as usual by $4\pi\epsilon_0\alpha = e^2/\hbar c$. This formula is correct to the extent that Z or $z \times (\alpha \approx 1/137)$ can be neglected relative to unity.]
(d.)
Under all of these conditions, using the results of (a.) and (c.) and working in the center of mass, show that the nonrelativistic elastic scattering result

$$\frac{d\sigma}{d\Omega} \propto \frac{1}{\sin^4\frac{\Theta}{2}}$$

does remain valid even when relativistic effects are taken into account.

20. (Relations used in particle physics)
(a.) Lorentz-invariant phase space [LIPS]
By transforming dp_x and E while keeping p_y and p_z fixed, show that

$$\frac{c\,dp_x}{E} dp_y dp_z$$

is invariant to a boost along x. (Since one can always define x to be the boost direction, and $d^3p \equiv dp_x dp_y dp_z$, a LIPS element

$$\frac{c\,d^3p}{E}$$

is invariant to a boost in *any* direction and therefore is Lorentz invariant.)
(b.)
Suppose a particle has momentum $\vec{p}$ and energy E. Define the particle's *longitudinal rapidity* y to be the boost along x that would be needed to make $p'_x = 0$ in the new frame $\mathcal{S}'$. Show that

$$y = \tanh^{-1}\frac{cp_x}{E} \, .$$

If your calculator doesn't have an arc hyperbolic tangent button, use the equivalent definitions

$$y = \tfrac{1}{2} \ln \frac{E/c + p_x}{E/c - p_x}$$
$$y = \ln \frac{E/c + p_x}{\sqrt{p_y^2 + p_z^2 + m^2 c^2}} \; .$$

(c.)
Using the fact that the rapidity (boost) is the additive parameter for the Lorentz transformation, and that y is defined to be a boost along $\hat{x}$, argue that an increment dy in longitudinal rapidity must be the same in two Lorentz frames that differ only by a relative boost along $\hat{x}$. Use this argument to conclude that

$$dy \, dp_y dp_z$$

is invariant to boosts along $\hat{x}$, as was the LIPS element

$$\frac{c\, d^3p}{E}$$

in part (a.). (In fact, these two expressions are equal.) Invariance of the longitudinal rapidity interval dy is a godsend for proton collider users. Since the proton's interacting constituents (quarks or gluons) carry only a variable fraction of the proton momentum, the center of mass (CM) of the colliding constituents is boosted along the beam direction by a variable amount (typically of order unity). However, the *difference* in longitudinal rapidity between any pair of emitted particles is unaffected by this unwelcome CM boost.)
(d.)
Define the *pseudorapidity* y_{pseudo} as the longitudinal rapidity that a particle would have if it were ultrarelativistic. Show that

$$y_{\mathrm{pseudo}} = \tanh^{-1}(\cos\theta) \; ,$$

where θ is the angle between the particle's direction and the x axis. (This is another godsend: if a particle is known to be ultrarelativistic, its longitudinal rapidity can be approximated by its pseudorapidity, which can be measured by knowing only the particle's direction.) An equivalent definition of pseudorapidity is

$$y_{\mathrm{pseudo}} = -\ln\left(\tan\tfrac{\theta}{2}\right) \; .$$